Vibration of Cylindrical Shells by Hybrid Finite Element Method

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Theme

IN a recent paper[1] a hybrid finite element procedure was outlined for elastodynamic analysis of continua and it was illustrated by examples from the theory of plate vibrations. In the present paper we extend the application of the hybrid procedure to vibration analysis of cylindrical shells.

Contents

The formulation is based on a variational principle which is the generalization (to dynamic problems) of the principle given by Pian[2] for elastostatic problems. In the special case of the harmonic vibrations with frequency ω the pertinent functional over a domain subdivided into n elements can be written as[1,3]

$$\pi_H = \Sigma_p \left[T_p(r_i) - \omega^2 U^*{}_p(\tau_{ij}) - \int_{sp} t_i \dot{u}_{bi}\, ds + \int_{s\tau p} \bar{t}_i \dot{u}_{bi}\, ds \right] \quad (1)$$

$$p = 1, 2, \ldots n$$

where the kinetic energy T is a function of momentum vector r_i and the complementary strain energy U^* is a function of impluse tensor τ_{ij} ($\dot{\tau}_{ij} \equiv$ stress tensor). The remaining terms denote the work obtained by the inner product of the boundary velocities u_{bi} and surface impulses t_i. In π_H both τ_{ij}, within the elements, and u_{bi}, on the boundaries of the element, are subject to variation. The variations of τ_{ij} must satisfy the equilibrium equations while those of u_{bi} must conform to conditions of interelement compatibility. Equilibrium along interelement boundaries and the conditions of compatibility within the elements tend to be satisfied a posteriori through the process of extremization.

In discretizing the functional π_H the surface velocities are written in terms of some nodal velocities $\dot{q}_i$ as

$$\{\dot{u}_b\} = [L]\{\dot{q}\} \quad (2)$$

The elements of matrix $[L]$ are the appropriate interpolation functions. The interpolation functions for the impulses τ_{ij} are written in two parts

$$\{\tau\} = [A]_1\{\beta\}_1 + [A]_2\{\beta\}_2 \quad (3)$$

The first part $[A]_1\{\beta\}_1$, when substituted into the equilibrium equation $\tau_{ij,j} = r_i$, will yield zeros on the right-hand side, i.e., this part furnishes the homogeneous solution of equilibrium equation. The second part, when substituted into the equilibrium equation will yield the appropriate interpolation functions for r_i. The interpolation functions for t_i are of course obtained through the equation $t_i = \tau_{ij} n_j$. Finally when π_H is discretized its variation can be expressed in terms of variations of β_i and those of $\dot{q}_j$. Since the β_i in one element are independent from those in the other elements, the coefficients of their variations can be equated to zero for stationary conditions of π_H. The vanishing of these coefficients establishes a set of relations between β_i and q_j by means of which β_i can be eliminated from π_H. Finally the elements are connected and the element nodal velocities $\dot{q}_j$ are transformed to a set of independent velocities of the connected system $\dot{q}_k^*$, in the global coordinate system. Then the stationary conditions of π_H can be obtained by equating the coefficients of q_k^* to zero. The resulting equations are

$$[M]^*\{\dot{q}\}^* = \{S\}^* \quad (4)$$

Equation (4) is the momentum impulse relation for the connected system. The matrix $[M]$ which is frequency dependent is, appropriately, called the dynamic mass matrix. When the prescribed impulses S_k^* vanish a nontrivial solution is possible when $\det [M]^* = 0$. The zeros of this determinant are associated with the natural frequencies of the composite system.

For the element developed the well-known equilibrium equations for thin cylindrical shells were employed.[4] The effects of rotary inertia and shear deformation as well as in-plane inertia were neglected. The impulses consisted of $[M_s M_\theta M_{s\theta} N_s N_\theta N_{s\theta} Q_s Q_\theta]$. For the first three quantities, namely the angular impulses, three (independent) complete cubic polynomials were assumed. The interpolation functions for the remaining five quantities; namely the inplane impulses and the impulses associated with the shears were derived from those of the angular impulses via the five equations of equilibrium. The matrix $[A]_2$ was so arranged as to allow a complete quadratic polynomial for the description of the momentum vector r_i.

Along the element boundaries a cubic polynomial was assumed for the velocity normal to the plane of the shell. This resulted in a quadratic polynomial for angular velocity tangential to the boundary. The angular velocity normal to the edge was assumed independently in the form of a linear polynomial. Likewise linear polynomials were empolyed for in-plane velocities. This scheme resulted in five generalized degrees of freedom at each node of the element—three translational and two rotational. The element developed was empolyed to analyze several examples[5] and two of these will be reported. The first is that of a complete cylinder freely supported at its ends. The first three natural frequencies of this cylinder are given in Table 1 alongside with the dimensions of the shell. For the fundamental mode, $\frac{1}{8}$th of the shell was analyzed using 9 elements and 52 degrees of freedom. For the second and third modes $\frac{1}{14}$th and $\frac{1}{18}$th of the shell were analyzed, respectively. In these cases six elements and 35 degrees of freedom were used. For purposes of comparison the exact solutions for this example are also given in Table 1, using the Goldenveiser and the simplified Donnell Shell equations. These results include the effect of in-plane inertia.

The next example is that of a clamped cylindrical panel. This example has been investigated extensively and it is usually referred to as the "Lockheed" panel. The first three natural frequencies of this example, as calculated and measured by

Table 1 First three natural frequencies of a simply supported cylinder[a]

Mode	Exact frequency Goldenveiser equations[11,12]	Exact frequency Donnell equations[12]	Present hybrid element
1	527.9	536	529
2	551.3	561	554.5
3	565.1	574	567

[a] Dimensions of the Shell in inches: length 40, radius 20, thickness 0.04.

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Table 2 First three natural frequencies of the Lockheed shell panel[a]

Ref.	Number of elements for the whole shell	Total degrees of freedom after boundary conditions	First frequency, Hz	Second frequency, Hz	Third frequency, Hz
8			814.0	940.0	1260.0
9		225	869.9	958.2	1288.2
		145	872.22	960.89	1309.0
6	2 × 2	36	984.0	990.7	1372.1
	3 × 3	84	881.27	68.37	1318.17
	4 × 4	156	872.06	959.91	1294.62
	6 × 6	372	869.56	957.56	1287.56
7	8 × 8	245		970.14	1537.30
10	8 × 12	462	890.0	937.0	1311.0
Present	3 × 4	30	938.0	1024.0	1394.0
Note	4 × 5	60	892.25	985.0	1349.0
	5 × 5	80	870.5	965.5	1329.0
	6 × 6	125	867.26	961.25	1312.5

[a] Dimensions of the Shell in inches: length 3, width 4, radius 30, thickness 0.013.

other authors is shown in Table 2 alongside with frequencies calculated by the present formulation. The analysis by Henshell et al.[8] is also based on the hybrid formulation. However, in this paper authors obtain a stiffness matrix from static considerations, i.e., the stresses within an element satisfy the static equations of equilibrium alone. A mass matrix is derived from displacement interpolation functions over *the entire element* as in the displacement formulations.

For the same degrees of freedom, the results obtained by the present hybrid formulation compare very favorably with those obtained by other formulations. However, some important points should be noted.

In the displacement formulation one can reduce the number of degrees of freedom of the composite system considerably by neglecting the in-plane inertia forces and subsequently condensing the resulting equations. The effect of neglecting the in-plane inertia forces upon the first few natural frequencies of the shell is generally quite negligible. However, the procedure requires the inversion of a matrix whose order is equal to the number of in-plane degrees of freedom of the composite system. Further, after condensation the equations of motion will generally loose their banded nature.

From a mathematical point of view the omission of in-plane inertia forces is equivalent to the introduction of some constraints. In the displacement formulations, these constraints, which are initially in terms of force quantities, must first be converted to displacements via the stiffness matrix before they can be satisfied. This however, cannot be done at the element level since the nodal displacements of the elements are not independent on account of the interelement compatibility requirements. Hence the constraints can be satisfied only after the elements have been assembled.

In the hybrid procedure, on the other hand, the impulses within the elements are allowed to vary independently and are not tied to the nodal velocities and so one can eliminate the in-plane inertia impulses without affecting the boundary velocities. This is done merely by satisfying the homogeneous equations of equilibrium for the in-plane motion of the element, i.e., in the hybrid procedure in-plane inertia impulses can be removed at the element level. Further the size of the dynamic mass matrix of an element will not alter whether one neglects or accounts for the in-plane and/or rotary inertia forces.

References

[1] Tabarrok, B., "A Variational Principle for the Dynamic Analysis of Continua by Hybrid Finite-Element Method," International Journal of Solids and Structures, Vol. 7, 1971.

[2] Tong, P. and Pian, T. H. H., "A Variational Principle and the Convergence of a Finite-Element Method based on Assumed Stress Distribution", International Journal of Solids and Structures, Vol. 5, 1969.

[3] Pian, T. H. H., "Hybrid Models," Proceedings of the International Symposium on Numerical Methods in Structural Mechanics, Univ. of Illinois, Sept. 1971.

[4] Flugge, W., *Stresses in Shells*, Springer-Verlag, Berlin, 1960.

[5] Gass, N., "Vibration of Shells of Revolution by a Hybrid Variayional Principle, M.A.Sc. thesis, Dept. of Mechanical Engineering, Univ. of Toronto.

[6] Olson, M. D. and Lindberg, C. M., "Dynamic Analysis of Shallow Shells with a Doubly-curved Triangular Finite Element," *Journal of Sound and Vibration*, Vol. 19, 1971.

[7] Henshell, R. D., Neale, B. K. and Warburton, G. B., "A New Hybrid Cylindrical Shell Finite Element," *Journal of Sound and Vibration*, Vol. 16, 1971.

[8] Deb Nath, J. M., "Dynamics of Rectangular Curved Plates," Ph.D. thesis, 1969, Univ. of Southampton.

[9] Webster, J. J., "Free Vibrations of Rectangular Curved Panels," *International Journal of Mechanical Sciences*, Vol. 10, 1968.

[10] Deb Nath, J. M. and Petyt, M., "Free Vibration of Doubly Curved Rectangular Plates Including the Presence of Membrane Stresses with Special Reference to Finite Element Technique," TR 22, 1969, Inst. of Sound and Vibration, Southampton, England.

[11] Voss, H. M., "The Effect of an External Flow on the Vibration Characteristics of Thin Cylindrical Shells," *Journal of the Aerospace Sciences*, Vol. 28, No. 12, 1961.

[12] Sodhi, D. S., "Application of Stress Functions to Dynamic Analysis of Plates and Shells," Ph.D. thesis, 1972, Dept. of Mechanical Engineering, Univ. of Toronto.